Quantum Mechanics of Integrable Spins on Coadjoint Orbits

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Abstract

We investigate classical integrable spins defined on the reduced phase spaces of coadjoint orbits of $G = SU(N)$ and study quantum mechanics of them. After discussions on a complete set of commuting functions on each orbit and construction of integrable spin models on the flag manifolds, we quantize a concrete example of integrable spins on SU(3) flag manifold in the coherent state quantization scheme and solve explicitly the time-dependent Schrödinger equation.

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I. INTRODUCTION

The classical formulation of non-relativistic spin degrees of freedom started with the work in Ref. [1] by describing them on the phase space of S^2. Then, in an independent development of classical isospin particles, the equations of motion for isospin particles in the presence of external gauge field were written down [2] and the Lagrangian [3] and Hamiltonian [4,5] formulations for the classical isospin were accomplished. Recently, the path integral quantization for the spin was proposed [6–9] and it was extended to formulations for the coherent state path integral [10] of generalized $SU(N)$ spin [11,12]. Especially in Ref. [12], the coherent state path integral of an integrable spin model on a flag manifold was performed and an exact result was obtained for a special case.

In this paper, we further develop the classical spin theory on the coadjoint orbits [13] of $G = SU(N)$ group each of which is a reduced phase space of generalized spin degree of freedom and study the quantum mechanics of integrable systems on them. We use the coherent state quantization method [14] as our main tool because coupled with geometric quantization method, it provide a natural way to quantize the system in terms of the time-dependent Schrödinger equation which can be expressed as a single component wave equation. This approach is to be contrasted to the quantization scheme with unitary representation of coadjoint orbits of compact groups [13] which deals, in general, multi-component equation.

Let us start by giving a brief summary of symplectic structure on the coadjoint orbits. The configuration space for the generalized spin degrees of freedom is a Lie group $G = SU(N)$. Consider the cotangent bundle $T^*G \cong G \times \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G}$ of the group G [15]. There is a natural symplectic group action on T^*G via

$$G \times (G \times \mathcal{G}^*) \longrightarrow G \times \mathcal{G}^*$$

$$(g, (h, a)) \mapsto (gh, a). \tag{1.1}$$

Let us define the moment map $\rho : T^*G \to \mathcal{G}^*$ via

$$< X, \rho(m) >= m\left(\frac{d}{dt}\Big|_{t=0}\exp tX \circ g\right) \tag{1.2}$$

where $X \in \mathcal{G}$ and $m \in T_g^*G$ is a linear map of $T_gG \to \mathbf{R}$. The orbit space $\rho^{-1}(x)/G_x$ is well defined and called a reduced phase space. Here, G_x is the stabilizer group of the point $x \in \mathcal{G}^*$. The above procedure is called a symplectic reduction. Furthermore, it can be shown that the reduced phase space may be naturally identified with the coadjoint orbit $\mathcal{O}_x \equiv G \cdot x \subset \mathcal{G}^*$ [15]:

$$\rho^{-1}(x)/G_x \cong G/G_x \cong G \cdot x. \tag{1.3}$$

It can be shown that the reduction can also be achieved by Dirac's constraint analysis and more detailed analysis and explicit examples can be found in Ref. [16].

Let us consider possible types of $SU(N)$ coadjoint orbits $\mathcal{O}_{\{n_1,n_2,\cdots,n_l\}} \equiv SU(N)/SU(n_1)\times \cdots \times SU(n_l) \times U(1)^{l-1}$. Here we have $\sum_{i=1}^{l} n_i = N$ and the rank of the subgroup $H \equiv SU(n_1) \times \cdots \times SU(n_l) \times U(1)^{l-1}$ is equal to $N-1$. It is well known that there is a natural symplectic structure on the coadjoint orbits of Lie group [13]. They also have the complex structure inherited from those of $SL(N,\mathbf{C})$ and $P_{\{n_1,n_2,\cdots,n_l\}}$ since $\mathcal{O}_{\{n_1,n_2,\cdots,n_l\}} = SL(N,\mathbf{C})/P_{\{n_1,n_2,\cdots,n_l\}}$, where $SL(N,\mathbf{C})$ is the complexification of $SU(N)$ and $P_{\{n_1,n_2,\cdots,n_l\}}$ is a parabolic subgroup of $SL(N,\mathbf{C})$ which is the subgroup of block upper triangular matrices in the $(n_1+n_2+\cdots+n_l) \times (n_1+n_2+\cdots+n_l)$ block decomposition of an element of $SL(N,\mathbf{C})$. Borel subgroup B_N corresponds $P_{\{1,1,\cdots,1\}}$. Together with the symplectic structure, they become Kähler manifolds. Let us assume that the symplectic two form is given in the local complex coordinate $(\bar{z},z)$ by the Kähler form

$$\omega = \sum_{i,j}\omega_{ij}dz_i \wedge d\bar{z}_j \tag{1.4}$$

where ω_{ij} can be expressed in terms of Kähler potential W by

$$\omega_{ij} = i\partial_i\bar{\partial}_jW, \tag{1.5}$$

Then the Poisson bracket can be defined via

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$$\{F, G\} = \sum_{i,k} \omega^{ki} \left(\frac{\partial F}{\partial z_k} \frac{\partial G}{\partial \overline{z}_i} - \frac{\partial G}{\partial z_k} \frac{\partial F}{\partial \overline{z}_i} \right) \tag{1.6}$$

where the inverse ω^{ki} satisfies $\omega_{ik} \omega^{kj} = \delta_i^j$.

The plan of the paper is as follows. In Section 2, we give a classical description of integrable spins on coadjoint orbits. A complete set of commuting functions on each orbit and construction of a concrete integrable system on the maximal orbit of $SU(3)$ group is given. In Section 3, we quantize the system by using the technique of coherent state and solve explicitly the time-dependent Schrödinger equation which is set up by geometric quantization method. Section 4 contains conclusion and discussions.

II. INTEGRABLE SPIN ON $SU(3)$ FLAG MANIFOLD

Let us discuss about integrable models on $\mathcal{O}_{\{n_1, n_2, \cdots, n_l\}}$. We denote F_a's the Hamiltonian functions associated with the vector fields T_a's generated by the generators X_a's, $[X_a, X_b] = f_{abc} X_c$:

$$T_a \rfloor \omega = dF_a, \tag{2.1}$$

and these F_a's satisfy the Poisson-Lie relations [15]:

$$\{F_a, F_b\} = f_{abc} F_c. \tag{2.2}$$

The f_{abc}'s are structure constants of the group G. To construct commuting functions on $\mathcal{O}_{\{n_1, n_2, \cdots, n_l\}}$, first note that there exist $N - 1$ commuting Hamiltonian functions $F_3, F_8, F_{15}, \cdots, F_{N^2-1}$ which is obvious from Eq. (2.2). These functions leave the maximal torus $T^{N-1} \subset \mathcal{O}_{\{n_1, n_2, \cdots, n_l\}}$ invariant. Let us denote rank n Casimir invariant of $su(m)$ algebra by $C_n(m)$. For example, $C_2(3) = \frac{1}{2}(F_1^2 + F_2^2 + \cdots + F_8^2)$ and $C_2(2) = \frac{1}{2}(F_1^2 + F_2^2 + F_3^2)$. Then, using the Eq. (2.2) again, it can be inferred [12] that the functions $C_p(q) - C_p(q-1)$, $2 \leq p \leq q = 2, 3, \cdots, N - 1$ are the other commuting functions. Here we define $C_p(q) = 0$ for $q < p$. So we have a set of commuting functions which is given by

$$F_3, F_8, F_{15}, \cdots, F_{N^2-1}, F_{pq} \equiv C_p(q) - C_p(q-1). \tag{2.3}$$

Note that in deriving the above set of commuting functions, we only used the Poisson-Lie relations (2.2) of $SU(N)$ symmetry on each orbit $\mathcal{O}_{\{n_1,n_2,\cdots,n_l\}}$ and the number of commuting functions is $N^2 - N$ which equals to the half of the maximal orbit. This makes us suspect that the commuting functions (2.3) are not independent on an arbitrary orbit. An example can be given in $\mathcal{O}_{\{N-1,1\}} = SU(N)/SU(N-1) \times U(1)$ which is the complex projective space $CP(N-1)$. The complete set of commuting functions is given by $F_3, F_8, F_{15}, \cdots, F_{N^2-1}$ and all the F_{pq}'s can be expressed in terms of $F_3, F_8, \cdots, F_{N^2-1}$. More explicitly, the Hamiltonian functions F_a's are given by [17,5]

$$F_a(z, \bar{z}) = im \sum_{I,K=0}^{N-1} \bar{Z}_I (X_a)_{IK} Z_K \tag{2.4}$$

where m is an integer and Z_0 and Z_i are given in terms of the complex coordinates z_i on $CP(N-1)$ as follows:

$$Z_0 = \frac{1}{\sqrt{1+|z|^2}}, \quad Z_i = \frac{z_i}{\sqrt{1+|z|^2}}. \tag{2.5}$$

It can be easily checked that in $SU(3)$ case, for instance, we have

$$F_{22} = C_2(2) = \frac{m^2}{72}(2 + \frac{\sqrt{12}}{m}F_8)^2. \tag{2.6}$$

So only F_3 and F_8 are independent commuting functions.

It seems to be a complicated matter to discuss about which commuting set of functions should be chosen. Instead we give a couple of examples. Let us first consider the orbit $\mathcal{O}_{\{n,1,\cdots,1\}}$. It seems natural to choose the following as the complete set of commuting functions: $F_3, F_8, \cdots, F_{N^2-1}, F_{pq}$ with the constraint $p \leq N - n$. $n = N - 1$ corresponds to $CP(N)$ orbit (we define $C_1(q) = 0$). $n = 1$ corresponds to the the maximal orbit $\mathcal{O}_{\{1,1,\cdots,1\}} = SU(N)/U(1)^{N-1}$ which is usually called a flag manifold. The set of independent commuting functions is given by the Eq. (2.3). For example, in $SU(4)$ case, we have six commuting functions $F_3, F_8, F_{15}, F_{22} = 1/2(F_1^2 + F_2^2 + F_3^2), F_{23} =$

$1/4(F_4^2 + F_5^2 + F_6^2 + F_7^2 + F_8^2)$, $F_{33} = d_{abc}F_aF_bF_c$, where d_{abc}'s are the symmetric structure constants of $su(3)$. This case corresponds to the so-called non-commutative integrability [18] in contrast to the Liouville integrable system of $CP(N-1)$. If we consider $\tilde{G} = SU(N-1) \times U(1)$ group action on $\mathcal{O}_{\{1,1,\cdots,1\}}$ and its algebra $\tilde{\mathcal{G}}$ generated by the Hamiltonian functions, they satisfy the criteria for the non-commutative integrability:

$$\dim \tilde{\mathcal{G}} + \operatorname{rank} \tilde{\mathcal{G}} = \dim \mathcal{O}_{\{1,1,\cdots,1\}}. \tag{2.7}$$

The level set $M = \{x \in M : F_i = c_i, \; X_i \in su(N-1) \times u(1)\}$ is a smooth $N-1$ dimensional torus T^{N-1}.

In the rest of the paper, we will give an explicit construction of an integrable model on the $SU(3)$ flag manifold $\mathcal{O}_{\{1,1\}}$. Then we will try to quantize the system by the use of coherent state quantization method. We consider the integrable system with F_3, F_8, and $C_2(2) = 1/2(F_1^2 + F_2^2 + F_3^2)$ in involution with Hamiltonian given by

$$H \equiv H(F_3, F_8, C_2(2)). \tag{2.8}$$

The well-known quantization of the above Hamiltonian is to pursue a procedure within the framework of geometry of coadjoint orbits [13] and Borel-Weil-Bott theory [19] or to calculate the unitary representations of $SU(3)$ group restricted to Cartan subgroup. We will pursue the coherent state quantization method [14] in this paper. One of the motivation could be that the coherent state path integral formulation of the generalized spins [10–12] exhibit many interesting features which the aforementioned quantization schemes do not have. More importantly in this paper, the motivation is that although the general results, for example, representation theory of the compact $SU(3)$, are well-known, we are interested in more concrete examples in which the time-dependent Schrödinger equation can be written down explicitly and its solutions can be looked into. We find that the coherent state method provides a convenient tool for this. We will restrict our Hamiltonian to be at most a quadratic function of F_3 and F_8 in which case the analysis becomes more transparent. More general cases could be handled in the same manner, although it could lead to a more complicated situation. Let us consider

$$H = \sum_{m,n=3,8} C^{mn} F_m F_n + \sum_{m=3,8} D^m F_m + \gamma \tag{2.9}$$

where C^{mn}'s and D^m's are constants. Note that the above system describes a type of generalized spinning tops. In order to find an explicit expression for the Hamiltonian given above, we have to coordinatize the flag manifold $\mathcal{O}_{\{1,1\}}$. The ideal choice for the explicit construction of the symplectic structure seems to be the Bruhat coordinatization [19]. According to Bruhat cell decomposition, the flag manifold $\mathcal{O}_{\{1,1\}}$ can be covered with six coordinate patches. The convenient thing about Bruhat cell decomposition is that the largest cell provides a coordinatization (z_1, z_2, z_3) of nearly all of the flag manifold missing only lower-dimensional subspaces.

The largest cell on $\mathcal{O}_{\{1,1\}} = SL(3, \mathbf{C})/B_3$ is represented as follows [20]:

$$[g_c(z)]_{B_3} = \left[\begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} \right]_{B_3} \mapsto (z_1, z_2, z_3) \tag{2.10}$$

with $g_c \in SL(3, \mathbf{C})$. Symplectic structure is given by the Kähler potential W which was calculated explicitly in terms of z_i's as follows [20]:

$$W = \log(1 + |z_1|^2 + |z_2|^2)^p (1 + |z_3|^2 + |z_2 - z_1 z_3|^2)^q \tag{2.11}$$

where p, q are integers for quantizable orbits. Using the symplectic structure (1.4) expressed in terms of W, we can calculate the Hamiltonian functions F_a's associated with the generators X_a's using Eq. (2.1) [12]. The functions F_3 and F_8 are given by

$$
\begin{aligned}
F_3 &= \frac{p}{2} \frac{2|z_1|^2 + |z_2|^2}{L_1} + \frac{q}{2} \frac{|z_2 - z_1 z_3|^2 - |z_3|^2}{L_2}, \\
F_8 &= \frac{\sqrt{3}p}{2} \frac{|z_2|^2}{L_1} + \frac{q\sqrt{3}}{2} \frac{|z_2 - z_1 z_3|^2 + |z_3|^2}{L_2},
\end{aligned} \tag{2.12}
$$

where we defined

$$L_1 = 1 + |z_1|^2 + |z_2|^2$$

$$L_2 = 1 + |z_3|^2 + |z_2 - z_1 z_3|^2. \tag{2.13}$$

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It is to be mentioned that we ignored some constants in the above F_3 and F_8 for convenience. These will be supplemented when the zero point energies are discussed in the quantization procedure. Let us express the Hamiltonian in terms of

$$Q_1 = F_3 + \frac{1}{\sqrt{3}} F_8 = p \frac{|z_1|^2 + |z_2|^2}{L_1} + q \frac{|z_2 - z_1 z_3|^2}{L_2}$$
$$Q_2 = F_3 - \frac{1}{\sqrt{3}} F_8 = p \frac{|z_1|^2}{L_1} - q \frac{|z_3|^2}{L_2}, \tag{2.14}$$

to achieve the notational simplicity and write the Hamiltonian (2.9) as

$$H = \sum_{m,n=1}^{2} A^{mn} Q_m Q_n + \sum_{m=1}^{2} B^m Q_m + \gamma \tag{2.15}$$

where A^{mn}'s and B^m's are some constants.

III. COHERENT STATE QUANTIZATION

To quantize the above system, we use the coherent state quantization method [14]. Let us define

$$|z\rangle = \sum_{i=1}^{3} \exp(z_i E_i)|0\rangle \tag{3.1}$$

where $z = (z_1, z_2, z_3)$, E_i's are the three positive roots and $|0\rangle$ is the highest weight vector corresponding to the geometry of $SU(3)/U(1) \times U(1)$ [14]. The normalization for Eq. (3.1) is chosen so that

$$\langle \bar{z}'|z\rangle = \exp W(\bar{z}', z) = (1 + \bar{z}_1' z_1 + \bar{z}_2' z_2)^p (1 + \bar{z}_3' z_3 + (\bar{z}_2' - \bar{z}_1' \bar{z}_3')(z_2 - z_1 z_3))^q. \tag{3.2}$$

The resolution of unity is expressed as

$$I = \int d\mu(\bar{z}, z) \exp(-W(\bar{z}, z))|z\rangle\langle\bar{z}|. \tag{3.3}$$

We note that this definition is different from the usual one [14] by the normalization factor $N = L_1^{-p} L_2^{-q} = \exp(-W)$. We have chosen this definition here because in the subsequent analysis, $\bar{z}$ and z can be treated independently [21] and also it enables one to choose the holomorphic (or anti-holomorphic) polarization.

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Our main interest lies in the evaluation of the propagator

$$G(\bar{z}'', z'; t) = \langle \bar{z}'' | e^{-i\hat{H}t} | z' \rangle \qquad (3.4)$$

which satisfies the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t} G(\bar{z}'', z'; t) = \hat{H} G(\bar{z}'', z'; t) \qquad (3.5)$$

in the coherent state quantization. The explicit differential operator form for the Hamiltonian which is necessary to set up the Schrödinger equation can be guessed from the geometric quantization method [22]. According to geometric quantization of classical phase space $\mathcal{O}_{\{1,1\}}$ with symplectic structure ω, we quantize classical observable $O \equiv O(F_a)$ in which the functions F_a's satisfy the Poisson-Lie algebra Eq. (2.2). The prequantum operators corresponding to the Hamiltonian functions $Q_m(\bar{z}, z)$'s are given by

$$\hat{Q}_m = -i\nabla_m + Q_m \qquad (3.6)$$

where $\nabla_m \equiv T_{Q_m} - iT_{Q_m}\rfloor\theta$ and θ is the canonical one form $\omega = d\theta$ expressed as

$$\theta = i\bar{\partial}W. \qquad (3.7)$$

We will be working in an anti-holomorphic polarization in which the differential operator $\hat{Q}_m = \hat{Q}_m(\bar{z})$ is given by

$$\hat{Q}_1(\bar{z}) = \bar{z}_1\frac{\partial}{\partial\bar{z}_1} + \bar{z}_2\frac{\partial}{\partial\bar{z}_2}, \ \ \hat{Q}_2(\bar{z}) = \bar{z}_1\frac{\partial}{\partial\bar{z}_1} - \bar{z}_3\frac{\partial}{\partial\bar{z}_3}. \qquad (3.8)$$

It can be checked that

$$< \bar{z}|\hat{Q}_m|z > \equiv < \bar{z}'|\hat{Q}_m|z > \Big|_{z'\to z} = \hat{Q}_m(\bar{z}') < z'|z > \Big|_{z'=z} = Q_m(\bar{z}, z) < \bar{z}|z > \qquad (3.9)$$

using the reproducing kernel of the flag manifold $\mathcal{O}_{\{1,1\}}$ given in Eq. (3.2). We note that Q_1 and Q_2 generate the following torus action [12]

$$Q_1 : (z_1, z_2, z_3) \mapsto (e^{i\theta_1}z_1, e^{i\theta_1}z_2, z_3), \quad Q_2 : (z_1, z_2, z_3) \mapsto (e^{i\theta_2}z_1, z_2, e^{-i\theta_2}z_3) \qquad (3.10)$$

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and Eq. (3.8) should contain zero point energies. One way of calculating these would be to try to find a representation of $\hat{F}_a$'s which satisfy the full $SU(3)$ algebra: $[\hat{F}_a, \hat{F}_b] = if_{abc}\hat{F}_c$. After a straightforward computation using the equation (3.6), for example, with $SU(3)$ Gell-Mann structure constants, we find that the correct representation is given by

$$\hat{Q}'_1(\bar{z}) = \hat{Q}_1(\bar{z}) - (\frac{1}{2} + \frac{1}{3\sqrt{3}})p - \frac{2}{3\sqrt{3}}q, \quad \hat{Q}'_2(\bar{z}) = \hat{Q}_2(\bar{z}) - (\frac{1}{2} - \frac{1}{3\sqrt{3}})p + \frac{2}{3\sqrt{3}}q. \quad (3.11)$$

Since our Hamiltonian Eq. (2.15) is a function of commuting $\hat{Q}'_m$'s, there is no normal ordering ambiguity. We are working in the Kähler polarization in which the Hilbert space is the anti-holomorphic sections of the Hermitian line bundle

$$\nabla_{\partial/\partial z_i} \Psi = 0. \quad (3.12)$$

Using (3.7), we get $\nabla_{\partial/\partial z_i} \Psi = \frac{\partial}{\partial z_i} \Psi = 0$ and we have an anti-holomorphic section. Also $G(\bar{z}'', z'; t)$ is a function of $\bar{z}''$ but not of z''. Substituting the explicit form of the differential operator $\hat{Q}'_1$ and $\hat{Q}'_2$ into the operator version of the Hamiltonian (2.15), we get the following Schrödinger equation:

$$i\frac{\partial}{\partial t}G(\bar{z}'', z'; t) = \left(\sum_{i,j=1}^{3} \alpha_{ij}\bar{z}''_i \frac{\partial}{\partial \bar{z}''_i}(\bar{z}''_j \frac{\partial}{\partial \bar{z}''_j}) + \sum_{i=1}^{3} \beta_i \bar{z}''_i \frac{\partial}{\partial \bar{z}''_i} + \gamma' \right) G(\bar{z}'', z'; t) \quad (3.13)$$

where $\alpha_{ij} = \sum_{m,n} A^{mn}a_{mi}a_{nj}$, $\beta_i = \sum_m (2A^{mn}d_n + B^m)a_{mi}$ and $\gamma' = \gamma + A^{mn}d_m d_n + B^m d_m$. Also, a_{ni} is a 2×3 matrix whose entries are given by $a_{1i} = (1, 1, 0)$, $a_{2i} = (1, 0, -1)$ and $d_1 = -(\frac{1}{2} + \frac{1}{3\sqrt{3}})p - \frac{2}{3\sqrt{3}}q$ and $d_2 = -(\frac{1}{2} - \frac{1}{3\sqrt{3}})p + \frac{2}{3\sqrt{3}}q$.

The solution to the above equation can be expressed as follows:

$$G(\bar{z}'', z'; t) = \sum_{n_1, n_2, n_3=0}^{\infty} D_{n_1 n_2 n_3} \prod_{i=1}^{3} (\bar{z}''_i)^{n_i} \exp\left\{ -i(\sum_{i,j=1}^{3} \alpha_{ij} n_i n_j + \sum_{i=1}^{3} \beta_i n_i + \gamma')t \right\} \quad (3.14)$$

where

$$D_{n_1 n_2 n_3} = \prod_{i=1}^{3} \frac{1}{n_i!} \left(\frac{\partial}{\partial \bar{z}''_i} \right)^{n_i} G(\bar{z}'', z'; 0) \Big|_{\bar{z}''_1 = \bar{z}''_2 = \bar{z}''_3 = 0} \quad (3.15)$$

and

$$G(\bar{z}'', z'; 0) = (1 + \bar{z}''_1 z'_1 + \bar{z}''_2 z'_2)^p (1 + \bar{z}''_3 z'_3 + (\bar{z}''_2 - \bar{z}''_1 \bar{z}''_3)(z'_2 - z'_1 z'_3))^q. \quad (3.16)$$

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For $\alpha_{ij} = 0$, the Eq. (3.14) sums into a closed expression

$$G = (1 + \bar{z}_1'' z_1' e^{i\beta_1 t} + \bar{z}_2'' z_2' e^{i\beta_2 t})^p (1 + \bar{z}_3'' z_3' e^{i(\beta_2 - \beta_1)t} + (\bar{z}_2'' - \bar{z}_1'' \bar{z}_3'')(z_2' - z_1' z_3') e^{i\beta_2 t})^q e^{-i\gamma' t}.$$

(3.17)

We mention that the same result except the zero point energies was obtained by an explicit evaluation of the coherent state path integral [12].

Finally, the inner product in the Hilbert space is defined as

$$< \Psi_1(t) | \Psi_2(t) > = \int d\mu(\bar{z}, z) \exp(-W(\bar{z}, z)) \bar{\Psi}_1(z, t) \Psi_2(\bar{z}, t)$$

(3.18)

using the Eq. (3.3). Then, the wave function in the anti-holomorphic polarization at arbitrary time is given by

$$\Psi(\bar{z}'', t) = \int d\mu(\bar{z}, z) \exp(-W(\bar{z}, z)) G(\bar{z}'', z; t) \Psi(\bar{z}, 0)$$

(3.19)

It is to be mentioned that the anti-holomorphic polarization was chosen for convenience. The holomorphic polarization is equally viable and the analysis can be carried out without much change.

IV. CONCLUSION

In conclusion, we discussed about classical spin degrees of freedom on the coadjoint orbits of $SU(N)$ group and performed coherent state quantization of the system. We also considered an explicit example with the case of the $SU(3)$ flag manifold, set up the Schrödinger equation by the technique of geometric quantization and found an explicit solution. As was stressed before, quantization aspects of the Hamiltonian (2.8) in terms of compact coadjoint orbit and representation theory is well-known. In this paper, a detailed formulation in the framework of the single component time-dependent Schrödinger equation was provided. The two approaches are connected by change of basis from irreducible unitary representation to the coherent state basis [23]. It would be interesting to work out the relation explicitly in $SU(3)$ case. It would also be interesting to extend the same quantization procedure to

other cases. Possible extensions would be the one to the coadjoint orbits of other groups including the non-compact ones and also to generalize to a system of many spins in which the Hamiltonian consists of all the differential operators $\hat{F}_a$'s instead of only $\hat{F}_3$ and $\hat{F}_8$. Finally, generalization to non-relativistic field theory and their quantization would be another interesting topic to pursue. These will be reported elsewhere.

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[23] See A. Perelomov in Ref. [13], for example, in $SU(2)$ case.